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COERCIVENESS OF THE DISCONTINUOUS INITIAL-BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS

BY

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ABSTRACT

In this paper, the mixed problem for parabolic equations is investigated with the discontinuous coefficient at the highest derivative and with non-standard boundary conditions. Namely, the boundary conditions contain values of the solution not only on the boundary points, but also on the inner points of the considered domain as well. Moreover, abstract functionals are involved in the boundary conditions. We single out a class of functional spaces in which coercive solvability occurs for the investigated problem.

1. Statement of the problem

Let us consider a parabolic equation

$$\frac{\partial u}{\partial t} + a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)u = f(t,x), \quad t \in [0,T], \quad x \in [-1,1]\backslash\{0\} \tag{1.1}$$

with functional, many-point conditions

$$\begin{cases} L_1 u(t,\bullet) = \alpha_1 u_x^{(m_1)}(t,-1) & +\beta_1 u_x^{(m_1)}(t,-0) \\ & + \sum_{k=1}^{n_1} \eta_{1k} u_x^{(m_1)}(t,x_{1k}) + T_1 u(t,\bullet) = 0, \\ L_2 u(t,\bullet) = \alpha_2 u_x^{(m_2)}(t,+0) & +\beta_2 u_x^{(m_2)}(t,1) \\ & + \sum_{k=1}^{n_2} \eta_{2k} u_x^{(m_2)}(t,x_{2k}) + T_2 u(t,\bullet) = 0, \end{cases} \tag{1.2}$$

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transmission conditions

(1.3) $L_y u(t,\bullet) = \delta_{y-2} u_x^{(m_y)}(t,-0) + \gamma_{y-2} u_x^{(m_y)}(t,+0) + T_y u(t,\bullet) = 0, \quad y = 3,4$

and initial condition

$$u(0,x) = u_0(x), \tag{1.4}$$

where $a(t,x) = a_1(t)$ at $(t,x) \in [0,T] \times [-1,0)$ and $a(t,x) = a_2(t)$ at $(t,x) \in [0,T] \times (0,1]$; for each $y = 1,\ 2,\ 3,\ 4$, $m_y = 0$ or 1; $x_{1k} \in (-1,0)$, $x_{2k} \in (0,1)$ are intermediate points; T_y are abstract linear functionals; $a_i(t)$, $b(t,x)$ and $f(t,x)$ are complex valued functions; all coefficients of the boundary and transmission conditions are complex numbers; $u_x^{(m_y)}(t,\pm 0)$ denotes $\lim_{x\to\pm 0} u_x^{(m_y)}(t,x)$.

Problems of the above type arise, as a rule, in problems of the theory of heat and mass transfer (see, for example, [5]), in diffraction problems (see, for example, [1]) and in a varied assortment of physical transfer problems.

2. The Cauchy problem for the differential equation in Banach space corresponding to the problem (1.1)–(1.4)

For any $t \in [0,T]$ denote by $A(t)$ a linear operator in Banach space $L_q(-1,\ 1)$, $1 < q < \infty$, with domain of definition

$$D(A) = D(A(t)) = \{u \in W_q^2(-1,0) \oplus W_q^2(0,1) |\ L_y u = 0,\ y = 1,\ 2,\ 3,\ 4\}$$

independent of $t \in [0,T]$, and with action law

$$A(t)u = a(t,x)u''(x) + b(t,x)u(x). \tag{2.1}$$

Then the problem (1.1)–(1.4) may be rewritten in the differential-operator form

$$u'(t) + A(t)u(t) = f(t), \quad u(0) = u_0, \tag{2.2}$$

where $u(t) = u(t,\bullet)$, $f(t) = f(t,\bullet)$ are functions with values in the space $L_q(-1,1)$ and $u_0(\bullet) = u_0 \in L_q(-1,\ 1)$. Here, by $W_q^m(-1,0) \oplus W_q^m(0,1)$ we denote the Banach spaces of functions $u(x)$ on $(-1,\ 1)$ belonging to $W_q^2(-1,\ 0)$ and $W_q^2(0,\ 1)$ in $(-1,\ 0)$ and $(0,\ 1)$, respectively, with the norm $\|u\|_{q,m} = \|u\|_{W_q^m(-1,0)} + \|u\|_{W_q^m(0,1)}$ where, as usual, $L_q(a,b)$ and $W_q^m(a,b)$ are the well-known Sobolev spaces.

For estimating the resolvent of operator $A(t)$ we shall investigate the corresponding boundary-value problem for ordinary linear differential equations.

3. The associated spectral problem

The following nonhomogeneous boundary-value problem for ordinary differential equations is called the associated spectral problem for the problem (1.1)–(1.4):

$$L(\lambda)u = a(x)u'' + b(x)u - \lambda u = f(x), \quad x \in [-1,\ 1]\backslash 0, \tag{3.1}$$

$$L_y u = g_y, \quad y = 1,\ 2,\ 3,\ 4, \tag{3.2}$$

where the boundary functionals L_y are defined by (1.2) and (1.3); $b(x)$ is a bounded measurable function on $[-1,\ 1]$; $a(x) = a_1$ at $x \in [-1, 0)$, $a(x) = a_2$ at $x \in (0, 1]$; $a_i \neq 0$ and g_y are some complex numbers.

Note that boundary value problems for ordinary differential equations with functional, many point conditions were first investigated by S. Ya. Yakubov (see, for example, [8, 9]).

First we study the principal part of the problem (3.1), (3.2). Namely, denoting

$$L_0(\lambda)u = a(x)u''(x) - \lambda u(x), \quad L_{y0}u = L_y u - \tilde{T}_y u, \quad y = 1,\ 2,\ 3,\ 4, \tag{3.3}$$

where $\tilde{T}_y u = T_y u + \sum_{k=1}^{n_y} \eta_{yk} u^{(m_y)}(x_{yk})$ for $y = 1,\ 2$ and $\tilde{T}_y u = T_y u$ for $y = 3,\ 4$, consider the pure differential problem

$$L_0(\lambda)u = f(x), \quad L_{y0}u = g_y, \quad y = 1,\ 2,\ 3,\ 4. \tag{3.4}$$

In this paper, everywhere the angle $(-\pi + \overline{\omega} + \varepsilon) < \arg\lambda < (\pi + \underline{w} - \varepsilon)$, where $\overline{\omega} = \max\{\arg a_1, \arg a_2\}$ and $\underline{w} = \min\{\arg a_1, \arg a_2\}$, is denoted by G_ε and the number θ is defined as

$$\theta = \alpha_1\beta_2 \begin{vmatrix} \gamma_1 & \gamma_2 \\ \delta_1(\sqrt{a_1})^{m_3-m_4} & \delta_2(-\sqrt{a_2})^{m_3-m_4} \end{vmatrix},$$

where $\sqrt{z}$ denotes $\sqrt{|z|}\exp(i\arg z/2), -\pi < \arg z \leq \pi$.

LEMMA 3.1: *Let $\theta \neq 0$. Then for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$, such that for all $\lambda \in G_\varepsilon$ for which $|\lambda| > R_\varepsilon$ the linear operator*

$$\tilde{L}_0(\lambda)\colon u \to (L_0(\lambda)u, L_{10}u, L_{20}u, L_{30}u, L_{40}u)$$

from $W_q^2(-1,0) \oplus W_q^2(0,1)$ onto $L_q(-1,1) \oplus C^4$ is an isomorphism and for these λ for the solution of problem (3.4) the following coercive estimate holds:

$$\sum_{k=0}^{2} |\lambda|^{1-k/2} \|u\|_{q,k} \leq C(\varepsilon)\Big(\|f\|_{q,0} + \sum_{y=1}^{4} |\lambda|^{(2-m_y-1/q)/2} |g_y|\Big), \tag{3.5}$$

where $C(\varepsilon)$ is a constant which depends only on ε.

Proof: Continuity of the operator $\tilde{L}_0(\lambda)$ is obvious. Let us prove that for indicated λ the operator $\tilde{L}_0(\lambda)$ has a continuous inverse $\tilde{L}_0^{-1}(\lambda)$. Let $f \in L_q(-1,1)$, $\lambda \in G_\varepsilon$ and $|\lambda|$ be sufficiently large. We seek a solution $u(x) = u(x,\lambda)$ of the problem (3.1), (3.2) in the form of the sum $u(x) = u_1(x) + u_2(x)$, where $u_1(x) = u_1(x,\lambda)$ is some solution of the equation $L_0(\lambda)u_1 = f(x)$ and $u_2(x) = u_2(x,\lambda)$ is a solution of the boundary value problem

$$L_0(\lambda)u_2 = 0, \quad L_{y0}u_2 = g_y - L_{y0}u_1, \quad y = 1,\ 2,\ 3,\ 4. \tag{3.6}$$

For a short exposition, denoting $I_1 = (-1,0)$, $I_2 = (0,1)$ and $\lambda = s^2$, consider the equations

$$a_y u''_{1y}(x) - s^2 u_{1y}(x) = f_y(x), \quad x \in \mathbb{R} = (-\infty,\infty), \quad y - 1,2, \tag{3.7}$$

where $f_y(x) = f(x)$ at $x \in I_y$ and$f_y(x) = 0$ at $x \notin I_y$. Applying the Fourier transformation

$$(F\psi)(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\sigma x}\psi(x)dx$$

to the equation (3.7) we have

$$(a_y(i\sigma)^2 - s^2)Fu_{1y} = Ff_y.$$

Since for $\lambda \in G_\varepsilon$, $\sigma \in R$ the estimate

$$\left|a_y(i\sigma)^2 - s^2\right| \geq C(\varepsilon)\left(|\sigma|^2 + |s|^2\right) \tag{3.8}$$

holds, by virtue of the well-known properties of the Fourier transformation for $k = 0,\ 1,\ 2$ we have

$$\begin{aligned} u_{1y}^{(k)}(x) &= F^{-1}\left((i\sigma)^k F u_{1y}\right) \\ &= F^{-1}\left((i\sigma)^k (a_y(i\sigma)^2 - s^2)^{-1} F f_y\right), \end{aligned} \tag{3.9}$$

where F^{-1} is the inverse Fourier transformation. In view of (3.8) for $\lambda \in G_\varepsilon$ the functions

$$h_{ky}(\sigma) = s^{2-k}(i\sigma)^k \left(a_y(i\sigma)^2 - s^2\right)^{-1}, \quad k = 0,\ 1,\ 2$$

are continuously differentiable on $\mathbb{R}$ and satisfy the inequalities

$$|h_{ky}(\sigma)| \leq C(\varepsilon), \quad |h'_{ky}(\sigma)| \leq C(\varepsilon)|\sigma|^{-1}.$$

Then by virtue of the Mikhlin–Schwartz theorem [3, p. 1181] the functions $h_{ky}(\sigma)$ are Fourier multipliers from $L_q(-1,1)$ to $L_q(-1,1)$, i.e.

$$\|F^{-1}(h_{ky}\cdot Ff)\|_{L_q(R)} \le C_{ky}\|f\|_{L_q(R)}$$

for all $f \in L_q(\mathbb{R})$, where C_{ky} are constants. Hence the function

$$u_{1y}(x) = F^{-1}\left((a_y(i\sigma)^2 - s^2)^{-1} F f_y\right)$$

found from (3.9) at $k = 0$ is a solution of equation (3.7) belonging to $W_q^2(\mathbb{R})$ and for which the estimates

$$(3.10)\quad |s|^{2-k}\|u_{1y}^{(k)}\|_{L_q(\mathbb{R})} = \|F^{-1}(h_{ky}\cdot Ff)\|_{L_q(\mathbb{R})} \le C_{ky}\|f\|_{L_q(\mathbb{R})}, \quad k = 0,\ 1,\ 2$$

are satisfied. Consequently, the function $u_1(x)$ defined by $u_1(x) = u_{11}(x)$ at $x \in I_1$ and $u_1(x) = u_{12}(x)$ at $x \in I_2$ is a solution of the equation $L_0(\lambda)u_1 = f$ and, by (3.10), satisfies the following estimate:

$$(3.11)\qquad \sum_{k=0}^{2} |\lambda|^{1-k/2}\|u_1\|_{q,k} \le C(\varepsilon)\|f\|_{q,0}.$$

Now let us prove that for any complex numbers g_y, $y = 1,\ 2,\ 3,\ 4$ the problem (3.6) has a unique solution $u_2(x) = u_2(x,\lambda)$ belonging to $W_q^2(-1,0)\oplus W_q^2(0,1)$ and satisfying the estimate (3.5).

The general solution of the equation $L_0(\lambda)u = 0$ can be represented in the form

$$(3.12)\qquad u_2(x) = \sum_{y=1}^{4} c_y u_{2y}(x)$$

where the solutions $u_{2y}(x)$ are defined by $u_{2y}(x) = \exp(\lambda\omega_y x)$ at $x \in \Omega_y$ and $u_{2y}(x) = 0$ at $x \notin \Omega_y$, where $\omega_1 = (\sqrt{a_1})^{-1}$, $\omega_2 = -(\sqrt{a_1})^{-1}$, $\omega_3 = (\sqrt{a_2})^{-1}$, $\omega_4 = -(\sqrt{a_2})^{-1}$, $\Omega_1 = \Omega_2 = I_1$, $\Omega_3 = \Omega_4 = I_2$. By substituting (3.12) into the boundary conditions of problem (3.6) we obtain a system for finding the coefficients c_y, $y = 1,\ 2,\ 3,\ 4$,

$$(3.13)\qquad \sum_{y=1}^{4} c_y L_{i0} u_{2y} = g_i - L_{i0}u_1, \quad i = 1,\ 2,\ 3,\ 4.$$

Below, in terms of $[M]$, where M is some complex number, we denote any function of the form $M + r(\lambda)$, where $r(\lambda) \to 0$ for $\lambda \in G_\varepsilon$ as $|\lambda| \to \infty$. Then, using the inequality

$$\mathrm{Re}((-1)^y \omega_y s) \le -\frac{\sin\varepsilon}{2}|\omega_y||s|, \quad \lambda \in G_\varepsilon, \tag{3.14}$$

we can obtain the following asymptotic formula for the determinant $\Delta(\lambda) = \det(L_{i0}u_{2y})$ of the system (3.13),

$$\Delta(\lambda) = \begin{vmatrix} (\omega_1 s)^{m_1}[\beta_1] & (\omega_{2s})^{m_1}e^{-\omega_2 s}[\alpha_1] & 0 & 0 \\ 0 & 0 & (\omega_3 s)^{m_2}e^{\omega_3 s}[\beta_2] & (\omega_4 s)^{m_2}[\alpha_2] \\ (\omega_1 s)^{m_3}\gamma_1 & (\omega_2 s)^{m_3}\gamma_1 & (\omega_3 s)^{m_3}\delta_1 & (\omega_4 s)^{m_3}\delta_1 \\ (\omega_1 s)^{m_4}\gamma_2 & (\omega_2 s)^{m_4}\gamma_2 & (\omega_3 s)^{m_4}\delta_2 & (\omega_4 s)^{m_4}\delta_2 \end{vmatrix}.$$

Using the inequality (3.14) it is easy to verify that

$$\Delta(\lambda) = \omega_2^{m_1}\omega_3^{m_2} s^{\sum_{y=1}^4 m_y} e^{-\omega_2 s} e^{\omega_3 s}[\theta]. \tag{3.15}$$

Since $\theta \ne 0$, then for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for all $\lambda \in G_\varepsilon$, $|\lambda| > R_\varepsilon$ we have $\Delta(\lambda) \ne 0$. Thus, for these λ the system (3.13) has a unique solution

$$c_y = \sum_{i=1}^4 \frac{\Delta_{yi}(\lambda)}{\Delta(\lambda)}(g_i - L_{i0}u_1), \quad y = 1,\ 2,\ 3,\ 4,$$

where $\Delta_{yi}(\lambda)$ is an algebraic complement of the (y,i)th element of the determinant $\Delta(\lambda)$. Hence

$$u_2(x) = \sum_{y=1}^4 \left(\sum_{i=1}^4 \frac{\Delta_{yi}(\lambda)}{\Delta(\lambda)}(g_i - L_{i0}u_1 \right) u_{2y}(x). \tag{3.16}$$

Now we can estimate the norms $\|u_2^{(m)}\|_{q,0}$ for $m = 0,\ 1,\ 2$. Using the inequality (3.14) from the explicit form of the algebraic complement $\Delta_{yi}(\lambda)$ we can obtain the following asymptotic equations in the form

$$\begin{cases} \Delta_{y1}(\lambda) = s^{m_2+m_3+m_4} e^{(\omega_1+\omega_2)s}[\theta_{y1}], \\ \Delta_{y2}(\lambda) = s^{m_1+m_3+m_4} e^{\omega_2 s}[\theta_{y2}], \\ \Delta_{y3}(\lambda) = s^{m_1+m_2+m_4}[\theta_{y3}], \\ \Delta_{y4}(\lambda) = s^{m_1+m_2+m_3}[\theta_{y4}], \quad y = 1,\ 2,\ 3,\ 4, \end{cases} \tag{3.17}$$

where θ_{yi} are complex numbers.

Then from the asymptotic expressions (3.15), (3.17) and inequality (3.14) it follows that

$$\left|\frac{\Delta_{yi}(\lambda)}{\Delta(\lambda)}\right| \cdot \|u_{2y}^{(m)}\|_{q,0} \le C(\varepsilon)|s|^{-m_y-1/q+m}, \quad \lambda \in G_\varepsilon, \quad |\lambda| > R_\varepsilon. \tag{3.18}$$

Hence, for any $m = 0,\ 1,\ 2$ we have

$$\|u_2^{(m)}\|_{q,0} \le C(\varepsilon)|s|^{-m_y-1/q+m} \sum_{y=1}^{4}(|g_y| + |L_{y0}u_1|). \tag{3.19}$$

With the estimate $|L_{y0}u_1| = |L_{y0}u_1(\cdot,\lambda)|$ for $\lambda \in G_\varepsilon$, $|\lambda| \to \infty$ we use the interpolation inequality [2, Theorem 3.10.4]

(3.20)

$$\max_{x\in[a,b]} |u^{(j)}(x)| \le C\left(\varepsilon^{1-\tau}\|u^{(m)}\|_{L_q(a,b)} + \varepsilon^{-\tau}\|u\|_{L_q(a,b)}\right), \quad u \in W_q^m(a,b),$$

where $0 \le j < m$, $\tau = (j+1/q)/m$, $\varepsilon > 0$ is a sufficiently small number, C is a constant, and $[a,b]$ is a finite segment. Setting $\varepsilon = |s|^{-2}$ and taking into account estimate (3.11) we get

$$\begin{aligned}|L_{y0}u_1(\cdot,\lambda)| &\le C(\varepsilon)\sum_{j=0}^{m_y}\left(|s|^{-2+j+1/q}\|u_1''\|_{q,0} + |s|^{j+1/q}\|u_1\|_{q,0}\right)\\ &\le C(\varepsilon)|s|^{-2+m_y+1/q}\|f\|_{q,0}, \quad \lambda \in G_\varepsilon.\end{aligned}$$

Substituting this estimate into (3.19) we get

$$|s|^{2-m}\|u_2^{(m)}\|_{q,0} \le C(\varepsilon)\left(\|f\|_{q,0} + \sum_{y=1}^{4}|s|^{2-m_y-1/q}|g_y|\right), \quad m = 0,\ 1,\ 2,$$

which is valid for $\lambda \in G_\varepsilon$, $|\lambda| > R_\varepsilon$. It follows from this and (3.11) that the estimate (3.5) holds for the solution $u(x,\lambda) = u_1(x,\lambda) + u_2(x,\lambda)$ to problem (3.4). Uniqueness of this solution follows from the uniqueness of the solution to problem (3.7). The lemma is proved. ∎

4. The Fredholm property of the main spectral problem

To prove the coerciveness of the main spectral problem (3.1), (3.2) we shall first establish the Fredholm property of this problem. We assume that the linear bounded operator A from the Banach space E into a Banach space F called a Fredholm, for the range of values $R(A) = \{Au|\ u \in E\}$ closed in F, $\operatorname{Ker} A = \{u \in E|\ Au = 0\}$ and $\operatorname{co\,ker} A = \{v' \in F'|\ v'(Au) = 0, u \in E\}$ are finite-dimensional subspaces in E and F', respectively, and $\dim\ker A = \dim\operatorname{co\,ker} A$, where F' is a dual to F (see, for example, [9]).

LEMMA 4.1: *Let the linear functionals T_y, $y = 1, 2, 3, 4$ be continuous in space $W_q^2(-1,0) \oplus W_q^2(0,1)$. Then the linear operator*

$$\tilde{L}(\lambda)\colon u \to (L(\lambda)u, L_1u, L_2u, L_3u, L_4u)$$

from $W_q^2(-1,0) \oplus W_q^2(0,1)$ to $L_q(-1,1) \oplus \mathbb{C}^4$ is Fredholm.

Proof: Let $\lambda_0 \in G_\varepsilon$ be any complex number. Defining the auxiliary linear operator $\tilde{L}_1(\lambda)$ by

$$D(\tilde{L}_1(\lambda)) = D(\tilde{L}(\lambda)), \quad \tilde{L}_1(\lambda)u = (b(x)u(x) + (\lambda_0 - \lambda)u(x), \tilde{T}_1u, \ldots, \tilde{T}_4u),$$

we can represent the operator $\tilde{L}(\lambda)$ in the form of the sum $\tilde{L}(\lambda) = \tilde{L}_0(\lambda_0) + \tilde{L}_1(\lambda)$. For $\lambda_0 \in G_\varepsilon$ sufficiently large in modulus, the operator $\tilde{L}_0(\lambda_0)$ from $W_q^2(-1,0) \oplus W_q^2(0,1)$ onto $L_q(-1,1) \oplus \mathbb{C}^4$ is an isomorphism by Lemma 3.1. Since the functionals $\tilde{T}_y$ are continuous in $W_q^2(-1,0) \oplus W_q^2(0,1)$ and the embedding $W_q^2(-1,0) \oplus W_q^2(0,1) \subset L_q(-1,1)$ is compact [7, p. 350], then the operator $\tilde{L}_1(\lambda)$ is compact for any $\lambda \in \mathbb{C}$. Now, it is enough to apply the theorem of Fredholm operator perturbation [4, p. 238] to the operators $\tilde{L}_0(\lambda_0)$ and $\tilde{L}_1(\lambda)$, from which we obtain that the sum $\tilde{L}(\lambda) = \tilde{L}_0(\lambda_0) + \tilde{L}_1(\lambda)$ is Fredholm. The thorem is proved. ∎

5. Coerciveness of the main spectral problem

Now, using Lemma 3.1 and Lemma 4.1 we can establish the coerciveness of the problem (3.1), (3.2).

THEOREM 5.1: *Let $\theta \neq 0$, and the functionals T_y are continuous in the space $W_q^{m_y}(-1,0) \oplus W_q^{m_y}(0,1)$, $y = 1, 2, 3, 4$. Then for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$, such that for all $\lambda \in G_\varepsilon$ for which $|\lambda| > R_\varepsilon$ the operator $\tilde{L}(\lambda)\colon u \to (L(\lambda)u, L_1u, L_2u, L_3u, L_4u)$ from $W_q^2(-1,0) \oplus W_q^2(0,1)$ onto $L_q(-1,1) \oplus \mathbb{C}^4$ is an isomorphism, and for those λ for the solutions of the problem (3.1), (3.2) the coercive estimate*

$$\sum_{k=0}^{2} |\lambda|^{1-k/2} \|u\|_{q,k} \leq C(\varepsilon) \left(\|f\|_{q,0} + \sum_{y=1}^{4} |\lambda|^{(2-m_y-1/q)/2} |g_y| \right) \tag{5.1}$$

holds.

Proof: First we shall establish the a priori estimate for the solutions of the problem (3.1), (3.2). It is clear that any $u \in W_q^2(-1,0) \oplus W_q^2(0,1)$ satisfies the problem

$$L_0(\lambda)u = L(\lambda)u - b(x)u, \quad L_{y0}u = L_yu - \tilde{T}_yu, \quad y = 1, \ldots, 4.$$

Applying Lemma 3.1 to this problem we have the inequality

$$
\begin{aligned}
\sum_{k=0}^{2} |s|^{2-k}\|u\|_{q,k} \le & C(\varepsilon)\Big(\|L(\lambda)u - b(x)u\|_{q,0} \\
& + \sum_{y=1}^{4} |s|^{(2-m_y-1/q)}|L_y u - \tilde{T}_y u|\Big) \\
\le & C(\varepsilon)\Big(\|L(\lambda)u\|_{q,0} + \sum_{y=1}^{4} |s|^{(2-m_y-1/q)}|L_y u| \\
& + \|b(x)u\|_{q,0} + \sum_{y=1}^{4} |s|^{(2-m_y-1/q)}|\tilde{T}_y u|\Big)
\end{aligned}
\tag{5.2}
$$

which is valid for all $\lambda \in G_\varepsilon$ sufficiently large in modulus, where $\lambda = s^2$. Obviously

$$
\begin{aligned}
\sum_{y=1}^{4} |s|^{(2-m_y-1/q)}|T_y u| &\le C\sum_{y=1}^{4} |s|^{(2-m_y-1/q)}\|u\|_{q,m_y} \\
&\le C|s|^{-1/q}\sum_{k=0}^{2} |s|^{2-k}\|u\|_{q,k}.
\end{aligned}
\tag{5.3}
$$

Denote

$$T_y^0 u = \sum_{k=1}^{n_y} \eta_{yk} u^{(m_y)}(x_{yk}).$$

Let $y = 2$; then $x_{yk} \in (0,1)$. Using, for example, [2, 2.8.3], we construct the function $\eta(x)$, which satisfies the following conditions:

$$\eta \in C_0^\infty(\mathbb{R}), \quad \eta(x) = \begin{cases} 1, & x \in [\delta, 1-\delta], \\ 0, & x \in [0, \frac{1}{2}\delta] \cup [1-\frac{1}{2}\delta, 1], \end{cases} \quad 0 \le \eta(x) \le 1,$$

where $\delta = \min_{y,k}\{x_{yk}, 1-x_{yk}\}$. Then

$$|T_y^0 u| \le C\|u^{(m_y)}\|_{C[\delta,1-\delta]} \le C\|(\eta u)^{(m_y)}\|_{C[0,1]}. \tag{5.4}$$

By virtue of [2, Theorem 3.10.4],

$$|s|^{2-m_y-1/q}\|u^{(m_y)}\|_{C[0,1]} \le C(\|u\|_{q,2} + |s|^2\|u\|_{q,0}). \tag{5.5}$$

From (5.4), (5.5) and estimate (3.5), for a solution of problem (3.4) for these λ we find that

$$\begin{aligned}|s|^{2-m_y-1/q}|T_y^0 u| &\le C|s|^{2-m_y-1/q}\|(\eta u)^{(m_y)}\|_{C(0,1)}\\ &\le C(\|\eta u\|_{q,2}+|s|^2\|\eta u\|_{q,0}) \le C(\varepsilon)\|L_0(\lambda)(\eta u)\|_{q,0}\\ &\le C(\varepsilon)\Big(\|\eta L(\lambda)u\|_{q,0}+\|\eta(x)b(x)u(x)\|_{q,0}+\sum_{k=1}^{2}|s|^{2-k}\|u\|_{q,k-1}\Big) \quad (5.6)\\ &\le C(\varepsilon)\Big(\|f\|_{q,0}+\sum_{k=1}^{2}|s|^{2-k}\|u\|_{q,k-1}\Big).\end{aligned}$$

For any $\delta > 0$ we have

$$\|u\|_{q,k-1} \le \delta\|u\|_{q,k} + C(\delta)\|u\|_{q,0};$$

then by virtue of (5.6)

$$\begin{aligned}|s|^{2-m_y-1/q}|T_y^0 u| &\le C(\varepsilon)\left[\|f\|_{q,0}+\sum_{k=1}^{2}|s|^{2-k}(\delta\|u\|_{q,k}+C(\delta)\|u\|_{q,0})\right]\\ &\le C(\varepsilon)\|f\|_{q,0}+(C(\varepsilon)\delta+C(\varepsilon,\delta)|s|^{-1})\sum_{k=0}^{2}|s|^{2-k}\|u\|_{q,k}. \quad (5.7)\end{aligned}$$

Here we used the following obvious inequality:

$$\sum_{k=1}^{2}|s|^{2-k}\|u\|_{q,0} \le |s|^{-1}|s|^2\|u\|_{q,0}.$$

Similarly, the estimate (5.7) can be established for the other values of y, i.e. $y = 1,\ 3,\ 4$. It is clear that for a fixed $\varepsilon > 0$ it is possible to choose $\delta > 0$ so small and $|\lambda|$ so large that

$$C|s|^{-1/q} + C(\varepsilon)\delta + C(\varepsilon,\delta)|s|^{-1} < 1.$$

By taking into account (5.3) and (5.7) in (5.2) we obtain for $|s| > R_\varepsilon$ the next a priori estimate,

$$\sum_{k=0}^{2}|s|^{2-k}\|u\|_{q,k} \le C(\varepsilon)\Big(\|L(\lambda)u\|_{q,0}+\sum_{y=1}^{4}|s|^{(2-m_y-1/q)}|L_y u|\Big). \quad (5.8)$$

From this inequality it follows that a solution of the problem (3.1), (3.2) is unique, i.e. $\dim\operatorname{Ker}\tilde{L}(\lambda) = 0$ and $\tilde{L}^{-1}(\lambda)$ is continuous. Then by virtue of Lemma 4.1, $R(\tilde{L}(\lambda)) = L_q(-1,1)\oplus\mathbb{C}^4$. Consequently, the operator $\tilde{L}(\lambda)$ is an isomorphism. The estimate (5.1) for the solutions of the problem (3.1), (3.2) is obtained from the found a priori estimate (5.8) immediately. Therefore, the theorem is proved. ∎

6. Coerciveness of the Cauchy problem for the discontinuous parabolic equation in the Banach space $L_q(-1,1)$

In this section, using the results of section 5 above, we indicate a class of Banach spaces for which the coercive solvability of the considered problem (1.1)–(1.4) has been undertaken.

By $C_0^\alpha([0,T],W_q^k)$ we shall define the set of continuous functions $u(t)$, on $[0,T]$, with values in Banach space $W_q^k(-1,0)\oplus W_q^k(0,1)$, $k=0,\ 1,\ 2$, for which the norm

$$\|u\|_{\alpha,(q,k)}=\max_{t\in[0,T]}\|u(t)\|_{q,k}+\sup_{0\le t<t+h\le T} t^\alpha\frac{\|u(t+h)-u(t)\|_{q,k}}{h^\alpha}$$

is finite, where $\alpha\in(0,1)$ is some number. It can be shown that the linear space $C_0^\alpha([0,T],W_q^k)$ is a Banach space under the above norm $\|u\|_{\alpha,(q,k)}$.

THEOREM 6.1: *Suppose the following conditions are satisfied:*

(1) $|\arg a_i(t)|>\pi/2$ *for any* $t\in[0,T]$, $i=1,\ 2$.

(2) *The functions* $t\to a_i$: $[0,T]\to\mathbb{C}$ $(i=1,2)$ *and* $t\to b(t,\bullet)$: $[0,T]\to L_q(-1,1)$ *satisfy the Hölder condition with power* α, *i.e. for all* $0\le t<t+h\le T$

$$|a_i(t+h)-a(t)|\le Ch^\alpha,\quad \|b(t+h,\bullet)-b(t,\bullet)\|_{q,0}\le Ch^\alpha.$$

(3) *For any* $t\in[0,T]$

$$\theta(t)=\alpha_1\beta_2\begin{vmatrix}\gamma_1 & \gamma_2\\ \delta_1(\sqrt{a_1(t)})^{m_3-m_4} & \delta_2(-\sqrt{a_2(t)})^{m_3-m_4}\end{vmatrix}\neq 0.$$

(4) *The functionals* T_y $(y=1,\ 2,\ 3,\ 4)$ *are continuous in the space* $W_q^{m_y}(-1,0)\oplus W_q^{m_y}(0,1)$.

Then for any $f\in C_0^\alpha([0,T],L_q)$ *and for any* $u_0\in W_q^2(-1,0)\oplus W_q^2(0,1)$ *such that* $L_y u_0=0$, $y=1,\ 2,\ 3,\ 4$ *there exists a unique solution* $u(t)=u(t,x)$ *of the problem (1.1)–(1.4) and the following coercive estimate holds:*

$$\left\|\frac{du}{dt}\right\|_{\alpha,(q,0)}+\|u\|_{\alpha,(q,2)}\le C(\|f\|_{\alpha,(q,0)}+\|u_0\|_{q,2}),\tag{6.1}$$

where the constant C *is independent of* f *and* u_0.

Proof: The problem (1.1)–(1.4) is transformed to the operator-differential form as (2.2). We set

$$\overline{\omega}=\sup_{t\in[0,T]}\max\{\arg(-a_1(t)),\arg(-a_2(t))\},$$

$$\underline{\omega}=\inf_{t\in[0,T]}\min\{\arg(-a_1(t)),\arg(-a_2(t))\}.$$

From the first condition of this theorem it follows that there exists a number $\varepsilon > 0$ so that $\{\lambda\colon \operatorname{Re}\lambda \geq 0\} \subset \{\lambda\colon -\pi + \overline{\omega} + \varepsilon < \arg\lambda < \pi + \underline{\omega} - \varepsilon\}$. Then by virtue of Theorem 5.1, there exist a number $\sigma > 0$ such that for the resolvent of the operator $A(t)$ the estimate

$$\|(A(t) + \lambda I)^{-1}\| \leq C(1 + |\lambda|)^{-1}, \quad \operatorname{Re}\lambda \geq \sigma \tag{6.2}$$

holds.

Consequently, there exists a number $\mu > 0$ such that the operator $A_\mu(t) = A(t) + \mu I$ has a bounded inverse.

Let us establish that the operator function $A(t)(A(0) + \mu I)^{-1}$ satisfies the Hölder condition with power α. Using inequality (3.20), in view of condition (2), for $u \in D(A)$ we have

$$\begin{aligned} \|(A(t+h) - A(t))u\|_{q,0} \leq & \|(a(t+h,\bullet) - a(t,\bullet))u''(\bullet)\|_{q,0} \\ & + \max_{x \in [-1,1]\setminus 0} |u(x)| \|b(t+h,\bullet) - b(t,\bullet)\|_{q,0} \\ \leq & Ch^\alpha \|u\|_{q,2}. \end{aligned} \tag{6.3}$$

It is necessary to note that, here and everywhere, any constant appearing in the estimates is denoted by the same symbol as C.

Further, by virtue of Theorem 5.1 the operator $A_\mu(0)$ acting from $W_q^2(-1,0) \oplus W_q^2(0,1)$ onto $L_q(-1,1)$ has a bounded inverse. Hence, the inequality (6.3) implies that

$$\|(A(t+h) - A(t))u\|_{q,0} \leq Ch^\alpha \|A_\mu(0)u\|_{q,0}, \quad u \in D(A).$$

From this, it immediately follows that

$$\|(A(t+h)A_\mu^{-1}(0) - A(t)A_\mu^{-1}(0))u\|_{q,0} \leq Ch^\alpha \|u\|_{q,0}, \quad u \in L_q(-1,1), \tag{6.4}$$

i.e. the operator function $A(t)A_\mu^{-1}(0)$ satisfies the Hölder condition with power α. Therefore, having in view the work [6] we get the following estimate for the solution of the problem (1.1)–(1.4):

$$\left\|\frac{du}{dt}\right\|_{\alpha,(q,0)} + \|(A + \mu I)u\|_{\alpha,(q,0)} \leq C(\|f\|_{\alpha,(q,0)} + \|u_0\|_{q,2}). \tag{6.5}$$

Now let us establish that the estimate

$$\|u\|_{\alpha,(q,2)} \leq C\|A_\mu u\|_{\alpha,(q,0)} \tag{6.6}$$

is valid for all $u \in C_0^\alpha([0,T], W_q^2)$ for which boundary conditions (1.2) and (1.3) hold. For this, first we establish a series of auxiliary estimates. By using the estimates (6.2) and (6.4) it follows that the real value function $t \to \|A_\mu(0)A_\mu^{-1}(t)\|$: $[0,T] \to \mathbb{R}$ is bounded. Then, having in view the latter conditions and using Theorem 5.1 we have

$$\begin{aligned} \|u\|_{q,2} \le C\|A_\mu(0)u\|_{q,0} &\le C\|A_\mu(0)A_\mu^{-1}(t)\| \cdot \|A_\mu(t)u\|_{q,0} \\ &\le C\|A_\mu(t)u\|_{q,0}, \quad u \in D(A). \end{aligned} \tag{6.7}$$

By using Theorem 5.1 for $0 \le t < t+h \le T$, we also have

$$\begin{aligned} \|u(t+h) - u(t)\|_{q,2} \le& C\|A_\mu(t)(u(t+h) - u(t))\|_{q,0} \\ \le& C(\|A_\mu(t+h)u(t+h) - A_\mu(t)u(t)\|_{q,0} \\ &+ \|(A_\mu(t) - A_\mu(t+h))u(t+h)\|_{q,0}). \end{aligned}$$

Further, recalling that the operator function $A_\mu(t)A_\mu^{-1}(0)$ satisfies the Hölder condition with power α, we get

$$\begin{aligned} \|(A_\mu(t) - A_\mu(t+h))u(t+h)\|_{q,0} &\le Ch^\alpha\|A_\mu(0)u(t+h)\|_{q,0} \\ &\le Ch^\alpha\|A_\mu(0)A_\mu^{-1}(t+h)\| \\ &\qquad \cdot \|A_\mu(t+h)u(t+h)\|_{q,0} \\ &\le Ch^\alpha\|A_\mu(t+h)u(t+h)\|_{q,0} \\ &\le Ch^\alpha \max_{0\le t\le T} \|A_\mu(t)u(t)\|_{q,0}. \end{aligned}$$

Hence,

$$\begin{aligned} \|u(t+h) - u(t)\|_{q,2} \le C(&\|A_\mu(t+h)u(t+h) - A_\mu(t)u(t)\|_{q,0} \\ &+ h^\alpha \max_{0\le t\le T} \|A_\mu(t)u(t)\|_{q,0}). \end{aligned} \tag{6.8}$$

Taking into account (6.7) and (6.8), we finally have

$$\begin{aligned} \|u\|_{\alpha,(q,2)} =& \max_{0\le t\le T} \|u(t)\|_{q,2} + \sup_{0\le t<t+h\le T} t^\alpha \frac{\|u(t+h) - u(t)\|_{q,2}}{h^\alpha} \\ \le& C\Big(\max_{0\le t\le T} \|A_\mu(t)u(t)\|_{q,0} + \sup_{0\le t<t+h\le T} t^\alpha \frac{\|u(t+h) - u(t)\|_{q,2}}{h^\alpha}\Big) \\ \le& C\Big(\max_{0\le t\le T} \|A_\mu(t)u(t)\|_{q,0} \\ &+ \sup_{0\le t<t+h\le T} t^\alpha \frac{\|A_\mu(t+h)u(t+h) - A_\mu(t)u(t)\|_{q,0}}{h^\alpha}\Big) \\ \le& C\|A_\mu u\|_{\alpha,(q,0)}. \end{aligned}$$

Putting this inequality in (6.5) we obtain the needed estimate (6.1). Thus the theorem is proved. ∎

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